Computable Elastic Distances Between Shapes

LAURENT YOUNES

Presented by Kexue Liu

11/28/00

Overview

- Introduction
- Comparison of plane curves
- Definition of distances with invariance restrictions
- The Case of closed curves
- Experiments

Introduction

- Why need measure distances between shapes?

  Human eyes can easily figure out the differences and similarities between shapes, but in computer vision and recognition, need some measurements(distance!)
- The definition of the distance is crucial.
- Base the comparison on the whole outline, considered as a plane curve.
- Related to snakes: Active contour model .
- Minimal energy required to deform one curve into another!

  It is formally defined from a left invariant Riemannian distance on an infinite dimensional group acting on the curves, which can be explicitly,easily computed.

Introduction

Principles of the approach

- From group action cost to deformation cost (distance between two curves)
- From distance on G to group action cost
- A suitable distance on G : which corresponds the energy to transform one curve to another.

Introduction

Principles of the approach

From group action cost to deformation cost

- $\mathfrak{C}$: the object space, each object in it can be deformed into another.
- The deformation is represented by a *group action*

  $G \times \mathfrak{C} \to \mathfrak{C}$, $(a,C) \to a.C$ on $\mathfrak{C}$

  i.e. for all $a,b \in G$, $C \in \mathfrak{C}$, $a.(b.C)=(ab).C$, $e.C=C$
- Transitive : for all $C_1, C_2 \in \mathfrak{C}$, there exist $a \in G$, s.t $a.C_1=C_2$.
- $\Gamma(a,C)$: the cost of the transformation $C \to a.C$

  (1) $$d(C_1, C_2) = \inf\{ \Gamma(a,C_1) , C_2 = a.C_1\},$$

  the smallest cost required to deform C_1 to C_2

Introduction

Principles of the approach

- Proposition 1:Assume G acts transitively on C and that Γ is a function defined on G **x** $\mathfrak{C}$, taking values in $[0,+\infty[$, s.t.

  i) for all $C \in \mathfrak{C}$, $\Gamma(e,C)=0$.

  ii) for all $a \in G$, $C \in \mathfrak{C}$, $\Gamma(a, C)=\Gamma(a^{-1},a.C)$

  iii) for all $a,b \in G$, $C \in \mathfrak{C}$, $a.C_1=C_2$,$\Gamma(ab,C) \leqslant \Gamma(b,C)+\Gamma(a,b.C)$

  then d defined by (1) is symmetric, satisfies the triangle inequality, and is such that $d(C,C)=0$ for $C \in \mathfrak{C}$

Principles of the approach

From distance on G to group action cost

- From elementary group theory, $\mathfrak{C}$ can be identified to a coset space on G, fix $C_0 \in \mathfrak{C}$, $H_0=\{h \in G,\ h.C_0= C_0\}$, $\mathfrak{C}$ can be identified to G/H_0 through the well-defined correspondence $a.H_0 \leftrightarrow a.C_0$

Proposition 2: Let d_G be a distance on G. Assume that there exists $\gamma: G \rightarrow \mathbb{R}$ s.t $\gamma(h)=1$ if $h \in H_0$ and, for all $a, b, c \in G$,

(2) $$d_G(ca, cb) = \gamma(c)\, d_G(a, b)$$

For $C \in \mathfrak{C}$, with $C=b.C_0$

(3) $$\Gamma(a, C) = d_G(e, a^{-1})/\gamma(b)$$

then Γ satisfies i), ii), iii) of proposition 1. The obtained distance is, $d(C, C') = \gamma(b)^{-1} \inf\{ d_G(e, a),\ aC' = C\}$, where $C=b.C_0$

Introduction

Principles of the approach

A suitable distance on G .

- If a = (a(t), $t \in [0,1]$) is such a path subject to suitable regularity conditions, we define the length L(a) and then set $d_G(a_0,a_1)=\inf\{L(a), a(0)=a_0, a(1)=a_1\}$
the infimum being computed over some set of *admissible* paths, d_G is symmetrical and satisfies the triangle inequality.
  - If a(t), $t \in [0,1]$ is admissible, so is a(1−t), $t \in [0,1]$ and they have the same length.
  - If a(.) and ã(.) are admissible, so is their concatenation, equal to a(2t) for $t \in [0,1/2]$ and to ã(2t−1) for $t \in [1/2,1]$ and the length of the concatenation is the sum of the lengths.

Introduction

Principles of the approach

- We define the length of the path by the formula

$$L(\mathrm{a}) = \int_0^1 \left\| \dot{\mathrm{a}}_t(t) \right\| dt$$

- For some norm. Thinking of $\dot{\mathrm{a}}_t(t)$ as a way to represent the portion of path between a(t) and a(t+ dt), defining a norm corresponds to defining the cost of a small variation of a(t). We must have

$d_G(\mathrm{a}(t),\mathrm{a}(t+dt)) = \gamma(\mathrm{a}(t))\, d_G(e, \mathrm{a}(t)^{-1}\mathrm{a}(t+dt))$

so the problem is to define γ and the cost of a small variation from the identity.

Comparison of the plane curves

Infinitesimal deformations

Energy Functional

- C = {m(s)=(x(s),y(s)), s∈[0,*l*] }

  The parametrization is done by arc-length, so we have: $\dot{x}_s^2 + \dot{y}_s^2 = 1$

  V(s)={u(s),v(s) } considered as a vector starting at the point m(s)

  $\tilde{C}$ = { $\tilde{m}$(s)=(x(s)+ u(s),y(s)+ v(s)}

  The field V (and its derivatives) is infinitely small.

  We define the energy of this deformation is :

$$\delta E^{(3)}(V) = \int_0^l \left\| \dot{V}_s(s) \right\|^2 ds$$

  $g^*_{\sim} : [0, l] -> [0, \tilde{l}\,]$ which associates s with the arc length $\tilde{s}$ in $\tilde{C}$ of the point m(s)+V(s)

Comparison of the plane curves

Infinitesimal deformations

- At order one, we have:

(5) $\dot{g}^*_s = 1 + \dot{u}_s \dot{x}_s + \dot{v}_s \dot{y}_s$

- Denote $\theta^*(s)$ the angle between tangent to C at point m(s) and the x-axis. Let $\tilde{\theta}^*(\tilde{s})$ be the similar function defined for $\tilde{C}$,we have $\cos(\theta^*) = \dot{x}_s$, $\sin(\theta^*) = \dot{y}_s$ and(at order one)

$$\cos\tilde{\theta}^* \circ g^* = (\dot{x}_s + \dot{u}_s)(1 - \dot{u}_s \dot{x}_s - \dot{v}_s \dot{y}_s) \approx \dot{x}_s - \dot{y}_s(-\dot{y}_s \dot{u}_s + \dot{x}_s \dot{v}_s)$$

$$\sin\tilde{\theta}^* \circ g^* = (\dot{y}_s + \dot{v}_s)(1 - \dot{u}_s \dot{x}_s - \dot{v}_s \dot{y}_s) \approx \dot{y}_s + \dot{x}_s(-\dot{y}_s \dot{u}_s + \dot{x}_s \dot{v}_s)$$

- Let $D^* = -\dot{y}_s \dot{u}_s + \dot{x}_s \dot{v}_s$, it is the normal component of $\dot{V}_s$.

at order one, we may write

Infinitesimal deformations

$$\cos\tilde{\theta}^* \circ g^* \approx \cos(\theta^* + D^*)$$

$$\sin\tilde{\theta}^* \circ g^* \approx \sin(\theta^* + D^*)$$

hence,

(6) $\tilde{\theta}^* \circ g^* - \theta^* = D^* = -\dot{y}_s \dot{u}_s + \dot{x}_s \dot{v}_s$

- $\delta E^{(3)} = \int_0^l (\dot{g}_s^* - 1)^2 ds + \int_0^l (\tilde{\theta}^* \circ g^*(s) - \theta^*(s))^2 ds$
- Set $g(s) = g(ls)/\tilde{l}, \lambda = \tilde{l}/l, \theta(s) = \theta^*(ls)$ and $\tilde{\theta}(\tilde{s}) = \tilde{\theta}^*(\tilde{l}\tilde{s})$

$$\delta E^{(3)} = l\int_0^l (\lambda \dot{g}_s - 1)^2 ds + l\int_0^l (\tilde{\theta} \circ g(s) - \theta(s))^2 ds \quad \text{(True Equality)}$$

- Set $D(s) = \tilde{\theta} \circ g(s) - \theta(s)$

$$\delta E^{(3)}(\lambda, g, D, l) = l\int_0^l [(\lambda \dot{g}_s - 1)^2 + D^2] ds$$

Comparison of the plane curves

Infinitesimal deformations

- Taking the first order, we have:

$$\delta E^{(3)}(\lambda, g, D, l) = l(\lambda - 1)^2 + l\int_0^l [(\dot{g}_s - 1)^2 + D^2]ds$$

- The functional involves some action on the curve C,

$(l, \theta(.))$ characterizes a curve up to translations.

- Define $\zeta(s) = \dot{x}_s(s) + i\dot{y}_s(s)$, it is a function from [0,1] to the unit circle of $\mathbb{C}$(denote it as Γ_1), then we may represent our set of objects as:
- (7) $\mathfrak{C}$={(l ,ζ) , l >0, ζ:[0,1]-> Γ_1,measurable}, ζ is translation and scale invariant.

Comparison of the plane curves

Infinitesimal deformations

- The transformation which can naturally be associated to λ, g, D:

  $$(l,\theta) -> (l\lambda, \theta \circ g^{-1} + D \circ g^{-1}) = (\tilde{l}, \tilde{\theta})$$

- Define the action:

  (9) $(\lambda, g, r).(l, \zeta) = (l\lambda, r.\zeta \circ g)$

  where $\lambda > 0$, g is a diffeomorphism of $[0,1]$ and r is a measurable function, defined on $[0,1]$ and with values in Γ_1.

- Let G be the set composed with these 3-uples, it is embedded product

  (10) $(\lambda_1, g_1, r_1).(\lambda_2, g_2, r_2) = (\lambda_1\lambda_2, g_2 \circ g_1, r_1.r_2 \circ g_1)$

  with identity $e_G = (1, \mathrm{Id}, 1)$ and inverse $(\lambda^{-1}, g^{-1}, \check{r} \circ g^{-1})$, $\check{r}$ is the complex conjugate of r.

Comparison of the plane curves

Infinitesimal deformations

- If (λ, g, r) is close to $(1, \mathrm{Id}, 1)$, then the energy functional is:

$$\delta E^{(3)}(\lambda, g, D, l) = l(\lambda - 1)^2 + l\int_0^l [(\dot{g}_s - 1)^2 + |r - 1|^2]ds$$

with first order approximation $|e^{-iD} - 1| = |D|$

- $a = (\lambda, g, r)$, we evaluate the cost of a small deformation C->a^{-1}C by $\Gamma(a^{-1}, C)^2 = l(\lambda - 1)^2 + l\int_0^l [(\dot{g}_s - 1)^2 + |r - 1|^2]ds$
- Let $C_0=(1,1)$, $b=(1/l, \mathrm{Id}, \zeta)$, then $C = (l, \zeta) = b.C_0$

$$\Gamma(a^{-1}, C)^2 = d_G(e, a)^2 / \gamma(b)^2$$

- With

(13) $\gamma(b) = 1/\sqrt{l}$

(14) $d_G(e, a)^2 = (\lambda - 1)^2 + \int_0^l [(\dot{g}_s - 1)^2 + |r - 1|^2]ds$

Comparison of the plane curves

Rigorous definition of *G*

- Consider Hilbert Space: $\mathcal{L}^2 = L^2([0,1], \mathbb{C})$
  with norm: $\|X\|_2^2 := \int_0^1 |X(s)| ds$
- Define
  (15) $$g^X(s) = \int_0^s |X(s)|^2 ds \Big/ \int_0^1 |X(s)|^2 ds$$
- Product
  (16) $$(X * Y)(s) = X(s) Y \circ g^X(s)$$
- It is well defined on

  (17) $$\tilde{G} = \{X \in \mathcal{L}^2, |X| > 0 \text{ almost everywhere}\}$$

Comparison of the plane curves

Rigorous definition of G

- Proposition 3 $\tilde{G}$ is a group for the operation $*$
- Proof:

(18) $\int_0^1 |X * Y|^2 ds = \int_0^1 |X|^2 ds \int_0^1 |Y|^2 ds$

Let the inverse of g^X is h , both of them are strictly increasing.

$$\int_0^s |X \circ h(v)|^{-2} dv = \int_0^{h(s)} |X(u)|^{-2} \left[|X(u)|^2 \Big/ \int_0^1 |X(v)|^2 dv \right] du$$

$$= h(s) \Big/ \int_0^1 |X(v)|^2 ds$$

Let $Y := 1/(X \circ h)$,then $\int_0^s |Y(v)|^2 dv = h(s) \Big/ \int_0^1 |X(u)|^2 du$

Comparison of the plane curves

Rigorous definition of G

thus $h = g^Y$ and $X * Y = Y * X = 1$

- For $X \in \tilde{G}, \lambda^X = \int_0^1 |X(s)|^2 ds, r^X = X^2 / |X|^2$
- Denote the mapping $\Phi : X -> (\lambda^X, g^X, r^X)$
- DEFINITION 1: Denote by G the set of 3-uples (λ, g, r) subject to the conditions:

1) $\lambda \in]0,+\infty[$

2) g is continuous on [0,1] with values in $\mathbb{R}$ and such that

- g(0)=0, g(1) = 1
- There exist a function q>0, a.e. on [0,1] such that $g(s) = \int_0^s q^2(\sigma) d\sigma$

3) r is measurable, $r : [0,1] -> \Gamma_1$, Γ_1 is the unit circle in $\mathbb{C}$

Comparison of the plane curves

Rigorous definition of G

- Proposition 4. $\Phi : \tilde{G} -> G$ is a group homomorphism.
- Remarks:
  - $\Phi(X) = \Phi(Y)$ iff $X^2 = Y^2$

    Denoting by $\mathcal{R}$ the equivalence relation $X^2=Y^2$

    then we can identity *G* with the quotient space $\tilde{G}/\mathcal{R}$

    The consistency of norm on $\mathcal{L}^2$ with formula (14).
  - Y=1+X, a small perturbation of 1 in $\mathcal{L}^2$, and assume |X(s)| is small for all s.

$$\lambda^Y - 1 \approx 2\int_0^1 \Re(X), \dot{g}_s^Y - 1 = |Y|^2 / \lambda^Y - 1 \approx 2\Re(X) - 2\int_0^1 \Re(X)$$

$$\text{and } r^Y - 1 = Y^2 / |Y|^2 - 1 \approx 2\Im(X), \text{so that}$$

$$(\lambda^Y - 1)^2 + \int_0^1 \left[(\dot{g}_s^Y)^2 + |r^Y - 1|^2 \right] \approx 4\int_0^1 |X|^2$$

Rigorous definition of *G*

thus $d_G(e,\Phi(Y))$ is identified for $Y \approx 1$ to $2\|Y-1\|_2$

Furthermore, d_G (e,a) for a close to e is the infimum of 2||Y-1||over all Y s.t Φ(Y) = a, which is the quotient distance on $\tilde{G}/\mathcal{R}$

- If $X \in \tilde{G}$, let $T_X : Y -> X * Y$ be the left translation on $\tilde{G}$.
  and T_X is linear and can be extended to all $Y \in \mathcal{L}^2$ and we have

$$\|T_X(Y)\|_2^2 = \int_0^1 |X|^2 \left|Y \circ \mathrm{g}^X\right|^2 ds = \lambda^X \int_0^1 |Y|^2 ds = \lambda^X \|Y\|^2$$

this means if $a = \Phi(X), b = \Phi(Y)$ with $X \approx Y$, From (13)

$$\gamma(a) = \sqrt{\lambda^X} \text{ and}$$

$$d_G(a,b) = \gamma(a) d_G(e, a^{-1}b) = 2\sqrt{\lambda^X} \left\|1 - X^{-1} * Y\right\|_2 = 2\|X - Y\|_2$$

Comparison of the plane curves

Admissible paths in *G*

- Definition 2:

A path $(X(t,.), t \in [0,1])$ is said to admissible in $\mathfrak{L}^2$ $(X(t,.) \in \mathfrak{L}^2$ for all $t)$ if there exists a path, denoted $(\dot{X}_t(t,.), t \in [0,1])$, s.t.

  - For all $\phi \in \mathfrak{L}^2$, the scalar function $t \to \int_0^1 X(t,s)\overline{\phi}(s)ds$ is differentiable in the general sense, and the derivative is $t \to \int_0^1 \dot{X}_t(t,s)\overline{\phi}(s)ds$
  - The total energy is finite: $\int_0^1 \int_0^1 \left|\dot{X}_t(t,s)\right|^2 dsdt < \infty$

Comparison of the plane curves

Admissible paths in *G*

- The length of an admissible path is:

$$\tilde{L}(X) = \int_0^1 \left[\int_0^1 \left| \dot{X}_t(t,s) \right|^2 ds \right]^{1/2} dt$$

- If a path is admissible in $\mathfrak{L}^2$ and $(\mathrm{t} \rightarrow X(t,s)) \in \tilde{G}$ for all t then it is admissible in $\tilde{G}$
- This definition satisfies the natural properties w.r.t. time reversal and concatenation.

Comparison of the plane curves

Admissible paths in *G*

- Definition 3: A path a(t), t∈[0,1] is admissible in *G* iff there exists a path *X*(t,.), t∈[0,1] which is admissible in $\tilde{G}$ and such that for all t, Φ(X(t,.))=a(t). We now define the length of a path a in *G* acting on C=(l,θ)(denoted $L_l[a]$) as $2\sqrt{l}$ times the length of a corresponding path in $\tilde{G}$.
- Proposition 5: If two admissible paths in $\mathcal{L}^2$,X(t,.) and Y(t,.), satisfy: $X(t,.)^2 = Y(t,.)^2$ for all t, then

Comparison of the plane curves

Invariant distance associated with *G*

- Compute the distance between two elements in *G* as the length of the shortest admissible path in *G* joining them.
- $d_G(a,b)=2\,\mathrm{Inf}\{||X-Y||_2,\ X,Y\in\check{G},\Phi(X)=a,\Phi(Y)=b\}$, the Inf is over all the shortest paths in Ğ joining X and Y.
- Paths of shortest length in $\mathcal{L}^2$ are straight lines but if $X,Y\in\check{G}$, the straight line $t\rightarrow tX+(1-t)Y$ does not necessarily stay within Ğ, however, the length of this straight line is $||X-Y||_2$. So we always have $d_G(a,b)\geqslant 2\min\{||X-Y||_2,\ X,Y\in\check{G},\Phi(X)=a,\Phi(Y)=b\}$.
- Equality is true provided the minimum in the right hand term is attainted for some X, Y such that $t\rightarrow tX+(1-t)Y$ stays in Ğ.

Comparison of the plane curves

Invariant distance associated with *G*

- Because $\|X-Y\|_2^2=\lambda^X+\lambda^Y-2\int_0^1\Re(X\overline{Y})$, the min is attained for $\Re(X\overline{Y})\geq 0$

- Theorem 1: One define a distance on *G* by

$$d_G^{(3)}(a,b)=2\left(\lambda+\mu-2\sqrt{\lambda\mu}\int_0^1\sqrt{\dot{g}_s\dot{h}_s}\left|\cos\left(\frac{\Delta-\tilde{\Delta}}{2}\right)\right|ds\right)^{1/2}$$

$$\text{for } a=(\lambda,g,e^{i\Delta}),\ b=(\mu,h,e^{i\tilde{\Delta}})$$

Comparison of the plane curves

Invariant distance associated with *G*

- One defines a distance between two plane curves by

$$d_G^{(3)}(C,\tilde{C}) = \left(l + \tilde{l} - 2\sqrt{l\tilde{l}} \sup \int_0^1 \sqrt{\dot{g}_s} \left| \cos\left(\frac{\tilde{\theta}\circ g(s) - \theta(s)}{2} \right) \right| ds \right)^{1/2}$$

for $C = (l, e^{i\theta})$, $\tilde{C} = (\tilde{l}, e^{i\tilde{\theta}})$.

the supremum being taken over functions g which are increasing diffeomorphisms of [0,1].

Comparison of the plane curves

Invariant distance associated with *G*

- The following lemma guarantee it is a distance!

- Lemma 1: one has

$$d^{(3)}(C,\tilde{C}) = 0 \Rightarrow l = \tilde{l} \text{ and } \theta = \tilde{\theta}$$

Definition of invariant

- Invariant: A distance d on $\mathfrak{C}$ is said to be invariant by a group of transformations Σ acting on $\mathfrak{C}$, if for all $\sigma \in \Sigma$, for all C_1, C_2 ,we have $d(\sigma C_1, \sigma C_2)= d(C_1,C_2)$.

- Weakly invariant: if there exists a function $\sigma \rightarrow q(\sigma)$ s.t. for all σ, C_1, C_2 , we have $d(\sigma C_1, \sigma C_2)= q(\sigma)d(C_1,C_2)$.

- Defined module Σ**:** If for all σ, C we have $d(\sigma C, C)= d(C, C)$.

  In the literature, the term ‘invariant’ is often used for the last definition.

Distances with invariance restrictions

Scale invariance

- By using appropriate energy definition and subspace of $\mathcal{L}^2$, we get a scale invariant distant.
- Theorem 2: One defines a distance on *G* by

$$d_G^{(2)}(a,b) = \left(\left|\log\lambda - \log\mu\right|^2 + 4\left(\arccos \int_0^1 \sqrt{\dot{g}_s \dot{h}_s} \left|\cos\left(\frac{\Delta - \tilde{\Delta}}{2}\right)\right| ds \right)^2 \right)^{1/2}$$

for a $= (\lambda, g, e^{i\Delta})$, b $= (\mu, h, e^{i\tilde{\Delta}})$

Distances with invariance restrictions

Scale invariance

- One defines a scale invariant distance between two plane curves by

$$d_G^{(2)}(C,\tilde{C}) = \left(\left|\log l - \log \tilde{l}\right|^2 + 4\left[\operatorname{Inf} \arccos \int_0^1 \sqrt{\dot{g}_s(s)} \left|\cos\left(\frac{\tilde{\theta} \circ g(s) - \theta(s)}{2} \right)\right| ds \right]^2 \right)^{1/2}$$

for $C = (l, e^{i\theta})$, $\tilde{C} = (\tilde{l}, e^{i\tilde{\theta}})$.

the infimum being taken over functions g which are strictly increasing diffeomorphisms of [0,1].

Distances with invariance restrictions

Modulo translations and scales

- Theorem 3: One defines a distance (modulo translations and scales) between two plane curves with normalized angle functions by:

$$d_G^{(1)}(C,\tilde{C}) = 2\mathrm{Inf}\ \arccos \int_0^1 \sqrt{\dot{g}_s(s)} \left| \cos\left(\frac{\tilde{\theta} \circ g(s) - \theta(s)}{2} \right) \right| ds$$

for $C = (l, e^{i\theta})$, $\tilde{C} = (\tilde{l}, e^{i\tilde{\theta}})$.

the infimum being taken over functions g which are strictly increasing diffeomorphisms of [0,1].

Distances with invariance restrictions

Modulo similarities

- Rotation invariance, the above distance is rotation invariant but not defined modulo rotation. Since rotations merely translate the angle functions.
- Theorem 4: one defines a distance by

$$d_G^{(0)}(a,b) = 2\min\left[\arccos\int_0^1 \sqrt{\dot{g}_s \dot{h}_s}\left|\cos\left(\frac{\Delta - \tilde{\Delta} - c}{2}\right)\right| ds\right]$$

for a $= (g, e^{i\Delta})$, b $= (h, e^{i\tilde{\Delta}})$, the min is c over $]-\pi, \pi]$

Distances with invariance restrictions

Modulo similarities

- One defines a distance(modulo similarities) between two plane curves with normalized angle functions by:

$$d_G^{(0)}(C,\tilde{C}) = 2\mathrm{Inf}\ \min\ \arccos \int_0^1 \sqrt{\dot{g}_s(s)} \left| \cos\left(\frac{\tilde{\theta} \circ g(s) - \theta(s) - c}{2} \right) \right| ds$$

for $C = (l, e^{i\theta})$, $\tilde{C} = (\tilde{l}, e^{i\tilde{\theta}})$, with inf over g, and min over $c \in]-\pi, \pi]$.

Case of closed curves

- Object set consists of all sufficiently smooth plane curves, including closed curves. So the distance we have used need to be valid for comparing closed curves, but modification is required.
- In case of open cures, there are only two choices for the starting points to parameterize the curves. But in closed curve case, the starting point maybe anywhere.
- Define $\tau_u.\zeta:[0,1]\text{->}\mathbb{C}$ s.t. $\tau_u.\zeta(s)= \zeta(s+u)$.

for $C_1=(l_1, \zeta_1)$ and $C_2=(l_2, \zeta_2)$, $d_c(C_1,C_2) = \inf d(\tau_u.C_1,C_2)$

where inf is over all u.

Experiments

- Database composed with eight outlines of airplanes for four types of airplanes.
- The shapes have been extracted from 3- dimentional synthesis images under two slightly different view angles for each airplane.
- Apply some stochastic noise to the outlines in order to obtain variants of the same shape which look more realistic.
- The lengths of the curves have been computed after smoothing(using cubic spline representation).

Experiments

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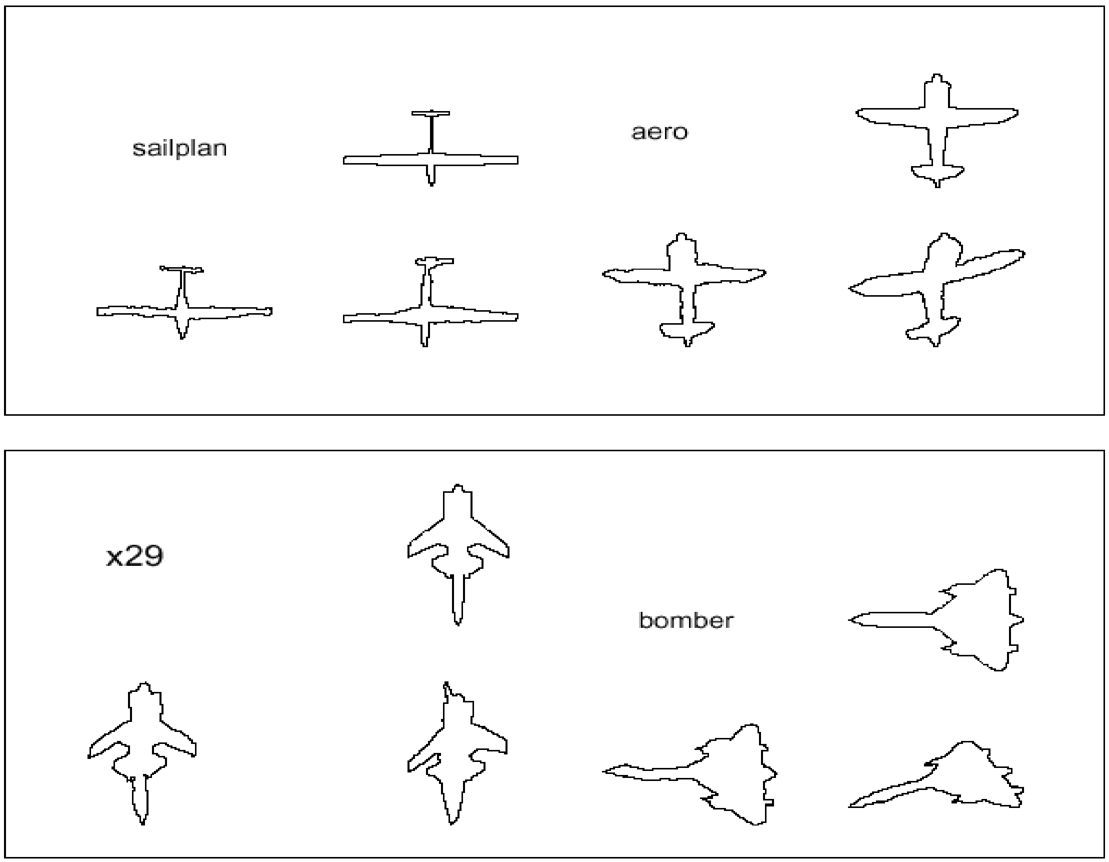

Fig. 1. *Outlines of planes from 4 classes. For each type of plane—upper right: original view (from above); lower left: view from above degraded by smooth noise; lower right: slight variation of the angle of view and noise. The compared outlines are lower left and lower right.*

Experiments

TABLE 1
Matrix of distances (in radian) within the plane database.

	sailp-1	sailp-2	aero-1	aero-2	x29-1	x29-2	bomb-1	bomb-2
sailp-1	0	0.25	0.43	0.46	0.79	0.73	0.9	0.81
sailp-2	0.25	0	0.47	0.48	0.71	0.69	0.77	0.82
aero-1	0.43	0.47	0	0.28	0.76	0.8	0.77	0.81
aero-2	0.46	0.48	0.28	0	0.79	0.77	0.78	0.76
x29-1	0.79	0.71	0.76	0.79	0	0.38	0.84	0.81
x29-2	0.73	0.69	0.8	0.77	0.38	0	0.82	0.8
bomb-1	0.9	0.77	0.77	0.78	0.84	0.82	0	0.29
bomb-2	0.81	0.82	0.81	0.76	0.81	0.8	0.29	0

cubic-spline representation). The complete matrix of distances has been computed on this database and is given in Table 1. We see that the distance between a plane and the other one from the same class is always smaller than between any plane in another